STRESS WAVE INTERACTION WITH A PERIODIC SYSTEM OF CURVILINEAR LONGITUDINAL SHEAR CRACKS

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Very high stress gradients occur in bulky members with initial crack-type defects under dynamic loads. If there are several cracks and they are relatively close together, then even for low levels of the external effects, they grow and, eventually, merge into a main crack. Hence, for the proven design of bulky constructions containing narrow cavities or slits it is necessary to have the solution of complex problems on elastic wave interaction with the defects. In a first approximation the total stress wave field can be separated into "plane strain" and "longitudinal shear" states. The simplest problem of stress wave interaction with a periodic system of curvilinear cracks under longitudinal shear conditions is studied in this paper. Such a formulation affords the possibility of taking account of the mutual influence of the cracks as well as their curvature on the dynamic stress intensity coefficient. The results obtained are approximately valid even in the case when the cracks are finite in number.

A description of the different stationary wave processes in an elastic medium with rectilinear cracks is contained in [1, 2] and the dynamic problem for a curvilinear crack in [3]. Results of investigations of elastic and electromagnetic wave diffraction by periodic structures are presented in [4-7].

1. Let us consider an unbounded elastic medium weakened by a 2L-periodic system of curvilinear slots tunneled along the Oz axis $l_j \equiv l \pmod{2L}$.

Let a monochromatic shear wave be radiated from infinity (w is the component of the elastic displacement vector along the Oz axis, ω is the circular frequency, and γ_2 is the wave number)

$$w_0 = \operatorname{Re}\{W_0 e^{-i\omega t}\},\ W_0 = \tau e^{-i\gamma_2 y},\ \tau = \text{const}, \tag{1.1}$$

and a load identical at congruent points and harmonic in time is given at the edges of the slots $X_n^{\pm} = Y_n^{\pm} = 0$, $Z_n^{\pm} = \operatorname{Re}\{Z^{\pm} e^{-i\omega t}\}$.

Under these conditions, a singular, 2L-periodic wave field of the longitudinal shear stresses τ_{xz}, τ_{yz} occurs in the medium.

Let D be the domain occupied by the medium, and let l be a simple open Lyapunov arc with beginning at the point a and terminus at b. Let us assume that $Z^+ = -Z^- = Z$ be a function of class H on the closed line $\bar{l}$ [8], where the plus sign refers to the left edge of l for motion from a to b (Fig. 1).

As is known, the determination of the wave field occurring in a medium with slots reduces to the boundary value problem

$$\nabla^2 W + \gamma_2^2 W = 0,\ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$
$$\frac{\partial}{\partial n}(W + W_0)^{\pm}|_{l_j} = \pm\frac{1}{\mu} Z^{\pm}\ (j = 0,\ \pm 1,\ \pm 2,\ \pm \ldots), \tag{1.2}$$

where μ is a shear modulus of the second kind, the upper sign corresponds to the value of the parenthesis on the left edge l_j, and differentiation is with respect to the direction of the positive normal to the left edge l_j.

We determine a 2L-periodic fundamental solution of the Helmholtz equation to construct the general representation of the solution of boundary-value problem (1.2). Let us start from the equation

$$\nabla^2 E + \gamma_2^2 E = \sum_{k=-\infty}^{\infty} \delta(x - 2kL,\ y), \tag{1.3}$$

where $\delta(x, y)$ is a two-dimensional Dirac δ function.

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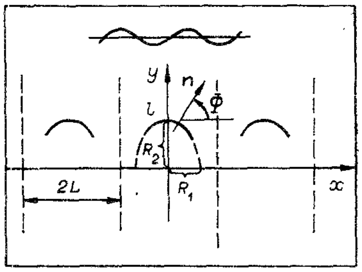

Fig. 1

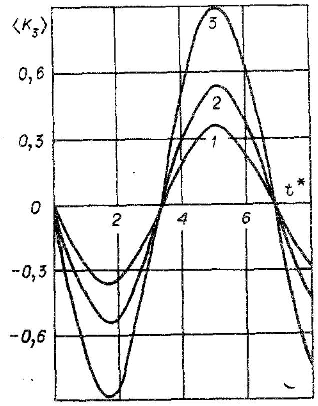

Fig. 2

Using the expansion of the 2L-periodic δ function

$$\delta_{2L}(x) = \sum_{k=-\infty}^{\infty} \delta(x - 2kL) = \frac{1}{2L} \sum_{k=-\infty}^{\infty} e^{i\alpha_k x}, \quad \alpha_k = \frac{\pi k}{L}, \tag{1.4}$$

separating variables in (1.3), and using the Fourier transform in the class D' of slow growth functions [9], we find

$$\begin{gathered} E(x, y) = \frac{1}{2L} \sum_{k=-\infty}^{\infty}{}' b_k(y)\, e^{i\alpha_k x}, \\ b_k(y) = -\frac{1}{2|r_k|} e^{-|r_k y|} \quad (k = \pm 1, \pm 2, \pm \ldots), \\ r_k^2 = \frac{\pi^2 k^2}{L^2} - \gamma_2^2, \quad \gamma_2 < \frac{\pi}{L}, \end{gathered} \tag{1.5}$$

where the prime on the sum denotes that the term with k=0 that yields a bounded solution of the Helmholtz equation and does not damp out as $y \to \infty$ should be discarded.

To extract the principal part of the fundamental solution we write the 2L-periodic fundamental solution E_0 of the highest operator in (1.3). By summing the appropriate series we obtain

$$\begin{gathered} E_0(x, y) = \frac{1}{2L} \sum_{k=-\infty}^{\infty}{}' a_k(y)\, e^{i\alpha_k x} = \frac{1}{4\pi} \ln \sin \frac{\pi z}{2L} \sin \frac{\pi \bar{z}}{2L} - \frac{|y|}{4L}, \\ a_k(y) = -\frac{1}{2|\alpha_k|} e^{-|\alpha_k y|}, \quad z = x + iy \quad (k = \pm 1, \pm 2, \pm \ldots). \end{gathered} \tag{1.6}$$

Taking account of (1.5) and (1.6), we represent the fundamental solution E in the final form

$$\begin{gathered} E(x, y) = E_0(x, y) + E^*(x, y), \\ E^*(x, y) = -\frac{1}{4L} \sum_{k=-\infty}^{\infty}{}' e^{i\alpha_k x} f_k(y), \\ f_k(y) = \frac{1}{|r_k|} e^{-|r_k y|} - \frac{1}{|\alpha_k|} e^{-|\alpha_k y|}. \end{gathered} \tag{1.7}$$

Therefore, the function E(x, y) defined in (1.7) satisfies the homogeneous Helmholtz equation at any point $z \neq 2Lk$ ($k = 0, \pm 1, \pm \ldots$,), possesses the characteristic singularities at the points of application of the concentrated functional, and damps out as $y \to \infty$.

After having extracted the principal part of the fundamental solution, the general term in the series in (1.7) damps out as k^{-3} at points of application of the concentrated functional (2Lk, 0).

2. We shall seek the solution of the boundary-value problem (1.2) in the form [3]:

$$W(x, y) = \frac{1}{2} \int_l P(\zeta) \left[\frac{\partial}{\partial z} E(z - \zeta)\, d\zeta - \frac{\partial}{\partial \bar{z}} E(z - \zeta)\, d\bar{\zeta} \right], \tag{2.1}$$

$$z = x + iy,\ \zeta = \xi + i\eta,\ z \in D,\ \zeta \in l,$$
$$\bar{z} = x - iy,\ \bar{\zeta} = \xi - i\eta,$$

where $P(\zeta)$ is the desired density, and $E(z-\zeta) = G(x-\xi, y-\eta)$ is determined by means of (1.7).

The representation (2.1) yields the 2L-periodic solution of the Helmholtz equation (1.2) that damps out as $y \to \infty$. Expanding the operator in (2.1), we obtain after manipulation

$$W(x, y) = \frac{i}{8L} \int_l P(\zeta)\,[g(z, \zeta) + G(z, \zeta)]\,ds,$$
$$g(z, \zeta) = \operatorname{Im}\left\{\operatorname{ctg}\frac{\pi(z-\zeta)}{2L}\frac{d\zeta}{ds}\right\}, \tag{2.2}$$
$$G(z, \zeta) = 2\sum_{k=1}^{\infty}[\alpha_k \cos\psi \sin\alpha_k(x-\xi) f_k(y-\eta)$$
$$-\sin\psi\cos\alpha_k(x-\xi) f_k'(y-\eta)\operatorname{sign}(y-\eta)] - \sin\psi\operatorname{sign}(y-\eta),$$

where $P(\zeta)$ is the desired function, $g(z, \zeta)$ is singular, $G(z, \zeta)$ is a bounded kernel, and ψ is the angle between the positive normal to the left edge l and the Ox axis.

Passing to the limit values in (2.2) as $z \to \zeta_0 \in l$, we find the jump in the displacement w on l

$$\Delta w = w^+ - w^- = 2\operatorname{Re}\{e^{-i\omega t}(W^+ - W^-)\} = \operatorname{Re}\{e^{-i\omega t}P(\zeta_0)\}. \tag{2.3}$$

There hence follows

$$P(a) = P(b) = 0. \tag{2.4}$$

On the basis of (2.4), we assume that $P(\zeta)$ is a function of class H [8] on the closed arc $\bar{l}$ that vanishes at its ends. We represent the boundary condition (1.2) in the form

$$e^{i\psi}\left[\frac{\partial}{\partial\zeta}(W + W_0)\right]^{\pm} + e^{-i\psi}\left[\frac{\partial}{\partial\bar{\zeta}}(W + W_0)\right]^{\pm} = \pm\frac{1}{\mu}Z^{\pm}. \tag{2.5}$$

Here the upper sign refers to the value of the quantity on the left edge of l during motion from a to b.

Evaluating the necessary derivatives of the function $W + W_0$, regularizing the divergent integral by integrating by parts [this operation is possible because of (2.4)], and then substituting the limit values of the derivatives in the boundary condition (2.5) as $z \to \zeta_0 \in l$, we arrive at the singular integrodifferential equation in the function $P(\zeta)$

$$\int_l P'(\zeta) h(\zeta, \zeta_0)\,ds + \int_l P(\zeta) H(\zeta, \zeta_0)\,ds = N(\zeta_0),$$
$$h(\zeta, \zeta_0) = \operatorname{Im}\left\{\cot\frac{\pi(\zeta_0 - \zeta)}{2L}e^{i\psi_0}\right\}, \qquad \zeta_0 \in l,$$
$$H(\zeta, \zeta_0) = 2\sum_{k=1}^{\infty}[\varphi_k(\zeta, \zeta_0)\cos\alpha_k(\xi_0 - \xi) + \sigma_k(\zeta, \zeta_0)\sin\alpha_k(\xi_0 - \xi)], \tag{2.6}$$
$$\varphi_k(\zeta, \zeta_0) = \alpha_k^2\cos\psi\cos\psi_0 f_k(\eta_0 - \eta) - \sin\psi\sin\psi_0 f_k''(\eta_0 - \eta),$$
$$\sigma_k(\zeta, \zeta_0) = \alpha_k\sin(\psi + \psi_0) f_k'(\eta_0 - \eta)\operatorname{sign}(\eta_0 - \eta), \quad \psi_0 = \psi(\zeta_0),$$
$$N(\zeta_0) = \frac{8L}{i\mu}Z(\zeta_0) + 8L\tau\gamma_2\sin\psi_0 e^{-i\gamma_2\eta_0}, \quad P'(\zeta) = \frac{dP(\zeta)}{ds}.$$

Here ds is an element of the arc l, the kernel h is singular, the kernel H is bounded, and the functions j_k are given in (1.7).

The function $P'(\zeta)$ has a singularity of the square-root type at the ends of the arc l, hence, to fix the solution it is necessary to attach an additional condition to (2.6). Because of (2.4) it has the form

$$\int_a^b P'(\zeta)\,ds = 0. \tag{2.7}$$

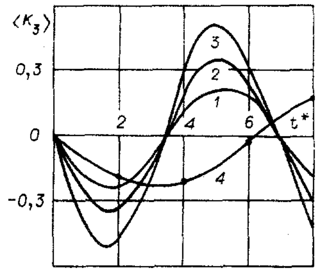

Fig. 3

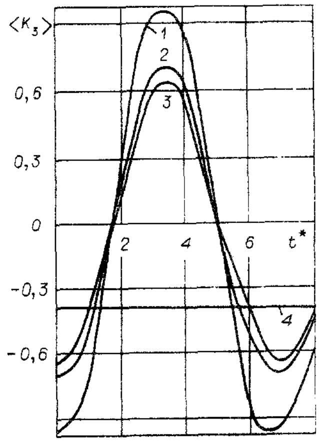

Fig. 4

We introduce parametrization of the contour l by the formulas

$$\zeta = \zeta(\beta),\ \zeta_0 = \zeta(\beta_0),\ -1 \leqslant \beta,\ \beta_0 \leqslant 1,\ a = \zeta(-1),\ b = \zeta(1) \tag{3.1}$$

for the numerical realization of (2.6) and (2.7).

In conformity with the above, we set

$$\begin{gathered} P(\zeta) = \Omega(\beta),\ P'(\zeta) = \Omega'(\beta)/s'(\beta), \\ \Omega'(\beta) = \Omega_0(\beta)/\sqrt{1-\beta^2},\ \Omega'(\beta) = d\Omega/d\beta,\ s'(\beta) = ds/d\beta, \end{gathered} \tag{3.2}$$

where $\Omega_0(\beta)$ is a function of class H on $\bar{l}$.

Taking account of (3.1) and (3.2), we represent Eqs. (2.6) and (2.7) in the form

$$\begin{gathered} \int_{-1}^{1} \Omega'(\beta)\left[\frac{1}{\beta-\beta_0} + K(\beta,\beta_0)\right] d\beta + \int_{-1}^{1} \Omega(\beta) R(\beta,\beta_0)\, d\beta = N_*(\beta_0), \\ \int_{-1}^{1} \Omega'(\beta)\, d\beta = 0, \quad K(\beta,\beta_0) = \operatorname{Re}\left\{\frac{\pi}{2L}\frac{d\zeta_0}{d\beta_0}\cot\frac{\pi(\zeta-\zeta_0)}{2L} - \frac{1}{\beta-\beta_0}\right\}, \\ R(\beta,\beta_0) = \frac{\pi}{L} s'(\beta_0)\, s'(\beta)\, H(\zeta(\beta),\zeta(\beta_0)), \\ N_*(\beta_0) = \frac{\pi}{2L} s'(\beta_0)\, N(\zeta(\beta_0)). \end{gathered} \tag{3.3}$$

The kernel $K(\beta, \beta_0)$ in (3.3) can possess no more than a weak singularity because of the assumptions relative to l, and the kernel $R(\beta, \beta_0)$ is bounded.

Using a procedure of Multhopp type [10], we reduce (3.3) to a system of linear algebraic equations in the values of the desired function $\Omega_0(\beta)$ at the Chebyshev nodes

$$\begin{gathered} \sum_{\nu=1}^{n} \alpha_{k\nu}\Omega_\nu^0 = f_k^* \quad (k = 1, 2, \ldots, n), \\ \alpha_{k\nu} = \frac{1}{\sin\theta_k}\cot\frac{\theta_\nu \mp \theta_k}{2} + K(\cos\theta_k, \cos\theta_\nu) - \frac{2}{n}\sum_{m=1}^{n} \sin\theta_m \cdot R(\cos\theta_m, \cos\theta_k)\,\delta_{\nu m}, \\ \delta_{\nu m} = \sum_{j=1}^{n-1} \frac{\cos j\theta_\nu \cdot \sin j\theta_m}{j}, \quad f_k^* = \frac{n}{\pi} N_*(\cos\theta_k), \\ \Omega_\nu^0 = \Omega_0(\cos\theta_\nu), \quad \theta_\nu = \frac{2\nu-1}{2n}\pi \quad (\nu = 1, 2, \ldots, n). \end{gathered} \tag{3.4}$$

Here the upper sign is taken in the case when $|k-\nu|$ is odd, while the lower is for when $|k-\nu|$ is even.

One of the equations of the system (3.4) must be discarded for the realization of the algorithm, and the additional condition

$$\sum_{\nu=1}^{n} \Omega_\nu^0 = 0$$

must be inserted instead.

4. The stresses in the medium are determined by the formulas

$$\tau_{xz} = \mu \partial w/\partial x, \ \tau_{yz} = \mu \partial w/\partial y. \tag{4.1}$$

Evaluating the appropriate derivatives, using (2.2), (3.2), and the nature of integrals of the Cauchy type in the neighborhood of the ends of the lines of integration [8], we obtain asymptotic formulas for the stresses in the neighborhood of the end of the slot c

$$\tau_{xz} - i\tau_{yz} = \frac{\mu \operatorname{Im}\{e^{-i\omega t}\Omega_0(\pm 1)\}}{4\sqrt{\mp 2\zeta'(\pm 1)}\sqrt{z-c}}, \tag{4.2}$$

where the upper sign corresponds to the vertex c = b, and the lower to c = a.

On the continuation beyond the crack vertex we obtain from (4.2)

$$\begin{gathered}\tau_{xz} = \frac{\mu \cos\psi(\pm 1)}{4\sqrt{2rs'(\pm 1)}} \operatorname{Im}\{e^{-i\omega t}\Omega_0(\pm 1)\}, \\ \tau_{yz} = \frac{\mu \sin\psi(\pm 1)}{4\sqrt{2rs'(\pm 1)}} \operatorname{Im}\{e^{-i\omega t}\Omega_0(\pm 1)\}, \\ \tau_n = \frac{\mu}{4\sqrt{2rs'(\pm 1)}} \operatorname{Im}\{e^{-i\omega t}\Omega_0(\pm 1)\}, \qquad r = |z-c|,\end{gathered} \tag{4.3}$$

where the upper sign refers to the terminus b, the lower to the beginning a, and τ_n is the stress on an area that is the continuation of the crack.

Therefore, exactly as in the static problem of longitudinal shear, crack propagation is possible only along a smooth trajectory. We determine the dynamic stress intensity coefficient from (4.3)

$$K_3 = \lim_{r\to 0}\left[\sqrt{2\pi r}\,\tau_n\right] = \frac{\mu}{4}\sqrt{\frac{\pi}{s'(\pm 1)}} \operatorname{Im}\{e^{-i\omega t}\Omega_0(\pm 1)\}.$$

Results of computations are presented below for a medium weakened by a periodic system of cracks along arcs of ellipses, whose parametric representation has the form (see Fig. 1)

$$x = R_1 \sin\beta\varphi, \ y = R_2 \cos\beta\varphi \ (-1 \leqslant \beta \leqslant 1).$$

1. Let the crack edges be force-free, and the monochromatic wave (1.1) be radiated from infinity, where

$$\tau = \tau_0 \exp(i\gamma_2 R_2).$$

Curves of the dynamic stress intensity coefficient $\langle K_3\rangle = K_3\sqrt{L}/\pi\mu\tau_0$ are presented in Fig. 2 as a function of the dimensionless time $t^* = \pi c_2 t/L$ (c_2 is the shear wave propagation velocity, and t is the time), the dimensionless wave number $\alpha = L\gamma_2/\pi = 0.9$, and the relative size of the domain $\xi_0 = \pi l/L = 0.2$; 0.4; 0.6 (curves 1-3, respectively) for $R_1 = R_2 = 0.5$, $\varphi = 45°$, $l = 0.392$.

The curves $\langle K_3\rangle$ are given in Fig. 3 in the same correspondence for $R_1 = 0.25$, $R_2 = 0.5$, $\varphi = 60°$, $l = 0.390$, $\alpha = 0.9$, $\xi = 0.2$; 0.4; 0.6. Here the curve 4 characterizing the change in $\langle K_3\rangle$ for a "straight" crack ($R_1 = 1$, $R_2 = 0.001$, $l = 0.174$) is given for $\alpha = 0.5$ and $\xi_0 = 0.2$. Appropriate results from [1] are superposed by points for comparison.

2. Let $\tau = 0$, and let a shear load harmonic in time and of constant intensity along the length of the crack be applied to the edges of the cracks. In this case $Z^+ = -Z^- = q$.

Graphs of the stress intensity coefficient $\langle K_3\rangle = K_3\sqrt{L}/\pi q$ are presented in Fig. 4 as a function of t^* for $\alpha = 0.9$, $\xi_0 = 0.2$; 0.4; 0.6 (curves 1-3, respectively) for cracks along the arcs of an ellipse $R_1 = 0.25$, $R_2 = 0.5$, $\varphi = 60°$, $l = 0.390$; curve 4 displays the static case $\alpha = 0$, $R_1 = 1$, $R_2 = 0.001$, $l = 0.174$, $\xi_0 = 0.2$.

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PLASTIC DEFORMATION UNDER A GENERALIZED PROPORTIONAL LOADING

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Plastic deformation is mainly the result of the displacement of one part of a crystal with respect to another. This latter specified the creation of physical theories of residual deformations within the framework of the slip concept [1]. On the basis of one such model, an attempt is made in [2, 3] to set up a connection between the stress and strain in time. To do this, a temperature–time operator was introduced into the governing relationships. The operator is introduced from the following physical considerations.

As is known, plastic flow in a material is developed extremely inhomogeneously and results in the appearance of local peak stresses [4-7]. According to [5], the peak stresses govern the resistance to plastic deformation to a significant degree. From an analysis of the experimental data [5-7], the deduction can be made that this stress microinhomogeneity, meaning also the resistance to plastic deformation, depends substantially on the loading and temperature modes. A rise in the loading rate and a reduction in the temperature result in an increase in the local peak stress fields, the appearance of significant elastic distortions of the crystal lattice. Such an increase in the microinhomogeneity results in an increase in the resistance to plastic deformation, as experiments show [4, 5].

However, the role of the peak stresses is not only to delay the development of plastic deformation. It follows from [6, 7] that the peak stresses exceeding the mean level are unstable and relax. This latter specifies numerous effects on the macrolevel: the relaxation of macrostresses, the delay in fluidity and creep, etc. The scalar measure, the temperature–time integral operator I, is taken as the microinhomogeneity characteristic of the stress state in a homogeneous continuous model of a solid. An approach to obtaining the operator I that is somewhat different from [2, 3] is proposed in this paper.

1. Let us represent an element of a polycrystalline body consisting of a large number of small particles in which the stresses are homogeneous and to which the mechanics of a continuous medium is applicable.

Let the stresses in particles at a specific time $t=s$ receive the increment

$$\Delta\sigma_{ij}(s) = \Delta\sigma^0_{ij}(s) + \Delta\sigma'_{ij}(s), \qquad (1.1)$$

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